PMM U.S.S.R.,Vol.54,No.3,pp.393-397,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

DETERMINATION OF THE DYNAMIC CONTACT STIFFNESS OF AN ELASTIC LAYER*

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An estimate of the limits of applicability of applied methods of computing the impedance characteristics of a medium that are utilized extensively in engineering practice is given on the basis of an exact solution of a dynamic contact problem. The dependence of the dynamic contact stiffness of a layer on the size of the contact domain and the frequency of the steady harmonic vibrations is analysed.

It is well-known that the unknown distribution under a stamp and the total reaction force of the medium /1/ must be found by solving an integral equation for a rigorous description of the interaction between a non-deformable stamp and an elastic medium (half-space or layer). Despite the numerous methods developed up to this time, the solution of the integral equations of dynamic contact problems is fairly time-consuming. Consequently, approximate approaches are often utilized in practice to determine the elastic reaction of the medium. For instance, the contact stress distribution is assumed to obey a certain law without solving the contact problem while the reaction of the medium (the compliance of the base) is determined by solving the first boundary-value problem of elasticity theory (the displacements are found for given surface stresses). In many cases this approach yields completely satisfactory results, but the question of the limits of its applicability is still open.

A comparison of the results of the exact and approximate approaches is presented below, a brief description is given of the methods used, and the most charactersitic results of computations are presented**. (**A detailed description of the derivation of the computational formulas, their realization on an electronic computer, and the numerical results obtained can be found in Glushkov E.V. and Glushkova N.V., Dynamic Reaction of an Elastic Layer; Comparison of the Exact and Approximate Approaches. Unpublished Paper 2250-B89, VINITI April 7, 1989.)

1. An elastic layer of thickness h is considered whose lower face is fastened stiffly to an undeformable base while a circular stamp of radius a vibrating under the action of a vertical load $Fe^{-i\omega t}$ is placed on the stress-free upper face. There is no friction between the stamp and the medium (smooth contact), and the stamp vibrations and points of the medium are assumed to be steady with angular frequency ω. The stamp mass is m and the layer characteristics are the density ρ, the shear modulus μ, and Poisson's ratio ν.

Henceforth all the dimensional quantities will be presented in units expressed in terms of $l_0 = h,\ \rho_0 = \rho,\ v_0 = \sqrt{\mu/\rho}$; for example $\omega = 2\pi l_0 f/v_0$, f is the frequency in Hertz.

The vertical translational displacements of the stamp are given by the expression

$$w = F/(P_1 - m\omega^2),\quad P_1 = \iint_\Omega q_1(x, y)\, dx\, dy = P/w \tag{1.1}$$

Here P is the total force acting on the surface of the medium in the contact domain, $q_1(x, y)$ is the stress distribution under the stamp during its vibration with unit amplitude, and Ω is the contact domain.

The function $P_1(\omega)$ describes the dynamic contact stiffness of the elastic medium. It is the reciprocal of the compliance function of the elastic foundation.

The stresses q_1 are determined from the integral equation /2/

$$\iint_\Omega k(x-\xi, y-\eta)\, q_1(\xi, \eta)\, d\xi\, d\eta = 1,\quad (x, y) \in \Omega \tag{1.2}$$

$$k(x, y) = \frac{1}{4\pi^2} \int_{\Gamma_1}\int_{\Gamma_2} K(\alpha)\, e^{-i(\alpha_1 x + \alpha_2 y)}\, d\alpha_1\, d\alpha_2$$

$$K(\alpha) = K_1(\alpha)/\Delta(\alpha),\quad K_1(\alpha) = {}^1/_2\varkappa_2^2\sigma_1(\sigma_1\sigma_2 \operatorname{sh}\sigma_1 \operatorname{ch}\sigma_2 - \alpha^2 \operatorname{sh}\sigma_2 \operatorname{ch}\sigma_1)$$

$$\Delta(\alpha) = 2\mu\,[2\alpha^2\sigma_1\sigma_2\delta + \alpha^2(\delta^2 + \sigma_1^2\sigma_2^2)\operatorname{sh}\sigma_1 \operatorname{sh}\sigma_2 - \sigma_1\sigma_2(\delta^2 + \alpha^4)\operatorname{ch}\sigma_1 \operatorname{ch}\sigma_2]$$

*Prikl.Matem.Mekhan.,54,3,474-479,1990

$$\delta = \alpha^2 - {}^1/_2\varkappa_2^2,\ \sigma_n = \sqrt{\alpha^2 - \varkappa_n^2},\ n = 1,\ 2,\ \alpha = \sqrt{\alpha_1^2 + \alpha_2^2}$$
$$\varkappa_1 = \omega/v_P,\ \varkappa_2 = \omega/v_S,\ v_P = \sqrt{(\lambda + 2\mu)/\rho},\ v_S = \sqrt{\mu/\rho}$$

The contours of integration Γ_1, Γ_2 coincide with the real axis, while deviating from it in the complex plane when circumventing real poles. The direction of the circumvention is dictated by the principle of limit absorption /2/.

Within the framework of the applied theory the contact stress distribution is not determined from the integral Eq.(1.2) but is assumed to obey a certain law $q(r),\ r = \sqrt{x^2 + y^2}$. The conditions of stamp contact with the layer are not satisfied here: displacements in the domain Ω

$$w(r) = \iint_\Omega k(x - \xi, y - \eta)\, q(\xi, \eta)\, d\xi\, d\eta = \frac{1}{2\pi} \int_\Gamma K(\alpha)\, Q(\alpha)\, J_0(\alpha r)\, \alpha\, d\alpha$$
$$\left(Q(\alpha) = \iint_\Omega q(r)\, e^{i(\alpha_1 x + \alpha_2 y)}\, dx\, dy = 2\pi \int_0^a q(r)\, J_0(\alpha r)\, r\, dr,\quad \alpha = \sqrt{\alpha_1^2 + \alpha_2^2}\right)$$

are not constant. Here $J_n(\alpha r)$ is the Bessel function.

The displacement of the surface of the medium

$$w_0 = \frac{1}{\pi a^2} \iint_\Omega w(r)\, d\Omega = \frac{1}{\pi a} \int_\Gamma K(\alpha)\, Q(\alpha)\, J_1(a\alpha)\, d\alpha$$

averaged over the contact domain is taken as a measure of the stamp displacement in this case.

Here

$$P_1 \approx \frac{P}{w_0},\quad P = 2\pi \int_0^a q(r)\, r\, dr \tag{1.3}$$

Henceforth two kinds of load are examined:
uniformly distributed

$$q(r) = \begin{cases} P/(\pi a^2), & r < a \\ 0, & r > a \end{cases},\quad Q(\alpha) = 2P\frac{J_1(a\alpha)}{a\alpha} \tag{1.4}$$

and having a parabolic distribution

$$q(r) = \begin{cases} P/(2\pi a \sqrt{a^2 - r^2}), & r < a \\ 0, & r > a \end{cases},\quad Q(\alpha) = \frac{P \sin a\alpha}{a\alpha} \tag{1.5}$$

The solution of the integral Eqs.(1.2) can be constructed by different methods in the case of a circular contact domain. The most efficient method is the method of factorization /2, 3/, by using which the problem can be reduced to an infinite linear algebraic system with exponentially decreasing off-diagonal coefficients. However, difficulties associated with the elimination of logarithmic singularities introduced by the Hankel function $H_0^{(1,2)}(a\alpha)$ for $\alpha = 0$ occur for the determination of $P_1 = Q_1(0)$. Consequently, the direct reduction of (1.2) to an infinite system with its subsequent regularization by the scheme developed in /4/ for static problems was used to perform the computations.

We have

$$Q_1(\alpha) = \frac{F(\alpha) + \Phi(\alpha)}{K(\alpha)},\quad F(\alpha) = \iint_\Omega e^{i(\alpha_1 x + \alpha_2 y)}\, dx\, dy = 2\pi\frac{J_1(a\alpha)}{\alpha} \tag{1.6}$$
$$\Phi(\alpha) = 2\pi \int_a^\infty \varphi(r)\, J_0(\alpha r)\, r\, dr$$
$$\varphi(r) = \frac{1}{2\pi} \int_\Gamma K(\alpha)\, Q_1(\alpha)\, J_0(\alpha r)\, \alpha\, d\alpha = \sum_{k=1}^\infty t_k H_0^{(1)}(\zeta_k r),\quad r > a$$
$$t_k = {}^1/_2\, i \operatorname{res} K(\alpha)\,|_{\alpha=\zeta_k} Q_1(\zeta_k)\, \zeta_k$$

where ζ_k are the poles $K(\alpha)$ located above the contour σ going along the real axis $\operatorname{Im}\alpha = 0$

and deviating from it only when circumventing the real poles in conformity with the limit absorption principle.

It follows from (1.6) that

$$\Phi(\alpha) = 2\pi a \sum_{k=1}^{\infty} \frac{t_k}{\alpha^2 - \zeta_k^2} [\zeta_k H_1^{(1)}(\zeta_k a) J_0(a\alpha) - \alpha H_0^{(1)}(a\zeta_k) J_1(a\alpha)] \tag{1.7}$$

The function $Q_1(\alpha)$ is entire; consequently, the poles induced by the zeros $K(\alpha)$ in the representation (1.6) should be eliminable, i.e., the condition

$$F(z_l) + \Phi(z_l) = 0, \; l = 1, 2, 3, \ldots \tag{1.8}$$

should be satisfied (z_l are the zeros of $K(\alpha)$), which yields a system to determine t_k

$$\mathbf{B}\cdot\mathbf{s} = \mathbf{g}; \; b = \| b_{lk} \|_{k, l=1}^{\infty} \tag{1.9}$$

$$\mathbf{s} = \{s_1, s_2, \ldots\}, \; \mathbf{g} = \{g_1, g_2, \ldots\}$$

$$B_{lk} = \frac{1}{z_l^2 - \zeta_k^2}\left[\zeta_k \frac{H_1^{(1)}(a\zeta_k) J_0(az_l)}{H_0^{(1)}(a\zeta_k) J_1(az_l)} - z_l\right], \quad g_l = -\frac{1}{z_l}, \quad t_k = \frac{s_k}{H_0^{(1)}(a\zeta_k)}$$

The condition

$$\Phi'(z_n) + F'(z_n) = 0$$

must be added to conditions (1.8) for double non-eliminable zeros z_n.

However, for even $\Phi(\alpha)$, $F(\alpha)$ this condition is satified identically for $z_n = 0$ and yields no additional equations in (1.9).

The elements of the matrix B have the behaviour

$$b_{lk} \sim (\zeta_k - z_l)^{-1}, \; l, k \to \infty$$

which permits extraction and inversion of the principal part of the system explicitly in the case when there are no multiple zeros.

The inverse matrix for the matrix $\mathbf{C} = \| c_{lk} \|_{l, k=1}^{\infty}$, $c_{lk} = 1/(\xi_k - z_l)$ is constructed in /4/.

Multiplying system (1.9) on the left by $\mathbf{C}^{-1}$ we arrive at a regularized system (E is the unit matrix)

$$(\mathbf{E} + \mathbf{R})\,\mathbf{s} = \mathbf{f}; \; \mathbf{R} = \mathbf{C}^{-1}\mathbf{D}, \; \mathbf{D} = \mathbf{B} - \mathbf{C}, \; \mathbf{f} = \mathbf{C}^{-1}\mathbf{g} \tag{1.10}$$

Having determined **s** from the system (1.10), we find

$$P_1(\omega) = Q_1(0) = P_F + P_\Phi = \frac{\pi a^2}{K(0)} - \frac{2\pi a^2}{K(0)} \sum_{k=1}^{\infty} s_k \frac{H_1^{(1)}(a\zeta_k)}{H_0^{(1)}(a\zeta_k)\,\zeta_k} \tag{1.11}$$

The quantity $P_F = F(0)/K(0)$ is identical with the solution of the unmixed problem for given surface displacements (unity in the domain Ω and zero outside the domain Ω), and $P_\Phi = \Phi(0)/K(0)$ takes account of the presence of displacements outside the domain Ω.

2. Graphs of the amplitude $|P_1|$, determined exactly by means of (1.11) (the solid line) and approximately for a uniform load (P_2 on the dashed line) and a parabolic load (P_3 on the dash-dot line) for $\nu = 0.3$ as a function of a and ω are shown in Figs.1 and 2. Where the results agree (in the scale of the sketch) only one line is presented. The dispersion curves (the graphs $\zeta_n(\omega)$, $z_n(\omega)$) for the case under consideration are in /2, 5/, while the form of the complex branches of these curves is presented in the paper mentioned in the footnote.

Analysis of the results shows that at low frequencies ($\omega < 3$) the approximate values P_2 and P_3 agree quite satisfactorily with the exact dependence of $P_1(a)$ in the range of variation $a \in [0.5]$ considered. For small a and $P_1(a)$ the curve $P_3(a)$ (a parabolic load) approaches more closely here while for large a the curve $P_2(a)$ (a uniform load) turns out to be closer in many cases.

The nearness of P_1 and P_3 for small a is explained by the fact that the principal terms of the asymptotic form of these functions as $a \to 0$:

$$P_m(a) = A_m \mu (1 - \gamma^2) a + O(a^2), \; a \to 0, \; m = 1, 2, 3 \tag{2.1}$$

$$A_1 = A_3 = 8, \quad A_2 = {}^3/_4\pi^2; \quad \gamma = v_S/v_P$$

(the quantity ω is fixed).

The representation (2.1) shows that for small a the dynamic contact stiffness is practically independent of the frequency and is identical with the static stiffness of a homogeneous half-space. This agreement is preserved over a fairly broad frequency range, excluding the neighbourhood of the natural vibrations frequencies of the layer.

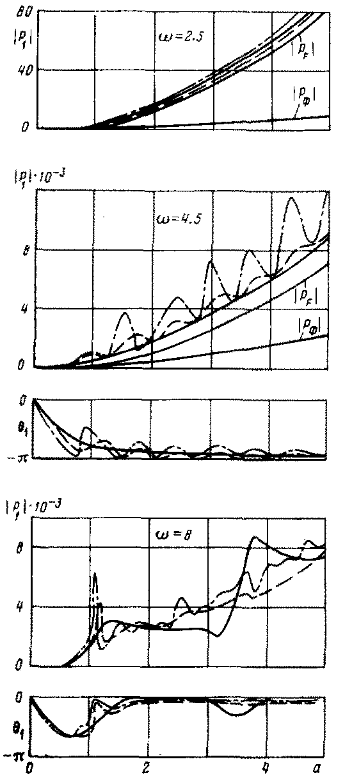

Fig.1

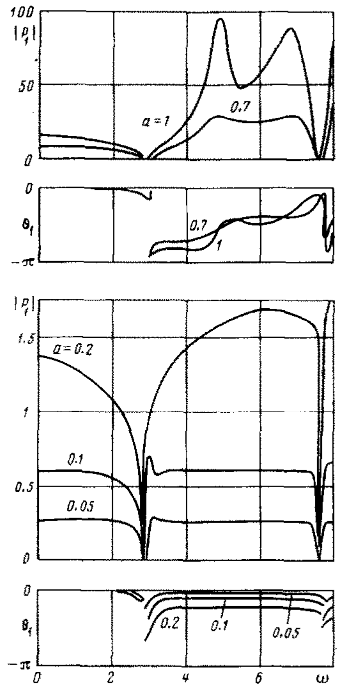

Fig.2

The agreement between the values of P_1 found by (1.11) and (2.1) for small a, with the requisite accuracy, was a good check on the method being used since it has constraints on a from below and its accuracy increases as a increases ($\|R\| \to 0$ in (1.10) as $a \to \infty$). Therefore, P_1 is determined with the requisite accuracy over the whole range of variation $a \in (0, \infty)$.

For $\omega > 3$ the curves of $P_2(a)$ and $P_3(a)$ are in agreement with the curve of $P_1(a)$ in a definite range $a < a_0(\omega)$ while periodically repeated deviations of the approximate curves from the exact one are observed for $a > a_0(\omega)$. The dependence of the upper bound for satisfactory agreement of the results a_0 on the frequency has the form of a hyperbola $a_0(\omega) \approx c/\omega$.

For $a < a_0$ the diameter of the contact domain is substantially smaller than the length of the surface waves being excited in the layer; consequently the layer surface is shifted entirely to one side of the equilibrium position under the action of a given harmonic load in the domain of its application. The average displacements w_0, determined by integrating the surface displacement in the domain of load application, describe the mean deviation of the surface from the equilibrium position in this case.

As ω increases the wavelength diminishes and buckling zones appear in the domain of load application. In this case the quantity w_0 decreases, resulting in an increase in P_2 and P_3 as compared with P_1.

Let us examine the change in phase θ_1 that governs the resonance properties of the medium during the vibration of massive bodies on its surface. For $\theta_1 = 0$ infinite resonances are possible, for small θ_1 bounded, and there are no resonances in the remaining cases /6/.

We have $\theta_1 \equiv 0$ in the quasistatic zone for any a $0 \leqslant \omega \leqslant \pi/2$ i.e., infinite resonances are always possible. But here the conditions for the origination of bounded resonances in the travelling wave zone $\omega > \pi/2$ depend substantially on ω and a. On the basis of an analysis of the relation $\theta_1(\omega)$ for a rectangular stamp of dimensions 3x4 a deduction was made earlier* (*Babeshko V.A., Glushkov E.V. and Glushkova N.V., On the Resonance Properties of a Stamp Elastic Layer System. Unpublished Manuscript 8329-B VINITI. December 4, 1985.) that the ranges in which bounded resonances $(\theta_1 \approx 0)$ are possible will alternate with those in which resonances cannot be $(\theta_1 \approx -\pi)$ and the range boundaries are the natural vibration frequencies of the layer (the frequency for the appearance of uneliminable double poles $\zeta_k = 0$). This deduction remains entirely valid for a commensurate with the layer thickness while $\theta \to 0$ for $a \to 0$ for all frequencies (Fig.1).

Therefore, the possibility of bounded resonances depends not only on ω but also on a: as $a \to 0$ a bounded resonance in mass is possible at any frequency $\omega > \pi/2$.

The study of the dependence of the contact stiffness on the frequency is of independent interest. Such results have been obtained earlier for large rectangular stamps (see /5/ and the bibliography given there). Curves of $|P_1(\omega)|$, $\theta_1(\omega)$ are presented in Fig.2 for different values of a.

As we noted above, the fact that for small a the quantity P_1 remains constant over a broad frequency range is essentially new. As before, $P_1 = 0$ at the layer natural vibration frequencies ($\omega = 2.89$; 2.93; 7.64; 8.82) including also for double $\zeta_k \neq 0$ on the left boundary of the reverse wave ranges (the dependence $P_1(\omega)$ in the reverse wave range was given earlier in a coarse scale*). (*Babeshko V.A., Glushkov E.V. and Glushkova N.V., On the Resonance Properties of a Stamp Elastic Layer System. Unpublished Manuscript 8329-B VINITI. December 4, 1985.)

The authors are grateful to V.A. Babeshko for discussing the results.

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Translated by M.D.F.

PMM U.S.S.R.,Vol.54,No.3,pp.397-402,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

PULSE PROPAGATION IN MEDIA WITH SMALL VELOCITY DISPERSION AND RELAXATION TIME SPECTRUM OF THE FORM $1/\tau$. EXACT SOLUTION*

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Pulse propagation in a medium whose dispersion-dissipative properties correspond to the presence of relaxation mechanisms in the medium, whose relaxation times form a spectrum of the form $g(\tau) \sim 1/\tau$, is considered. In the case of small velocity dispersion an exact solution is obtained for the pulse shape and it is shown that it is equivalent to the description of pulse propagation in a medium with "*Ei*-memory".

Acoustic wave propagation in real media can often be considered within the framework of a linear approximation of hereditary elasticity theory (HET) /1/. Phenomenological HET coefficients can be obtained using the theory of internal parameters /2/ characterized by relaxation times to a thermodynamic equilibrium state.

Exact solutions are known only for certain rheological models of media: a standard body /3/ characterized by a single relaxation time, its limiting case of Voigt /4, 5/ and Maxwell /6, 7/ media; in the case of small velocity dispersion for the model of a medium with "*E*-memory

**Prikl.Matem.Mekhan.*,54,3,480-484,1990